Post-Inflationary Reheating and Perturbations of the Cosmic Microwave Background.*

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Abstract

We formulate a gradual dynamical transition from a power-law inflation era with a scalar field to a radiation era with no scalar field including inhomogeneous perturbations to the Friedmann- Robertson-Walker universe. We show that for the cosmic microwave background radiation fluctuations this is excellently approximated by a sudden transition, with application of the Lichnerowicz conditions, both for density and gravitational wave perturbations.

I. INTRODUCTION

Post-inflationary reheating in which the energy of a scalar field of the inflationary era is converted into the relativistic particles of the radiation era is of intrinsic interest, and there may prove to be tests of the mechanisms of inflation and reheating. A common scenario is that there is a period of inflation followed by reheating, consequently followed by the radiation era; density and graviton perturbations are supposed to be of quantum origin arising in the inflaton field(s) early in the inflationary era. It is an interesting question how the reheating affects the perturbations of various wavelengths, particularly the density perturbations. For those wavelengths λ observable in the variations of the CMBR the phase $\Delta t/\lambda$, where Δt is the reheat time, is extremely small. So the usual assumption is that for these wavelengths the effect of the transition on the perturbations can be well calculated by the sudden (instantaneous) approximation. In recent years there has been some discussion on this point [1–5].It is the main purpose of this paper to support the usual assumption by illustration using a well known inflationary scenario followed by a particular reheating mechanism.

The mechanism that we use is an old one [6], that of a friction term in the scalar field equation with the corresponding balancing term in the radiation fluid energy equation. We

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do not postulate this as a close representation of the actual physical process of reheating or defrosting, which is anyway not yet determined, but use it as a stand-in providing a balanced transfer of energy-momentum from one phase to another in a mixed phase state. In this latter respect it is more physically motivated than reheating representations which just interpolate the scale factor of the FRW universe [1,3,4].

The new feature is that the calculation of the transition from a universe of all scalar field to one of all radiation fluid is done explicitly for the perturbed as well as for the unperturbed case. In the former case the synchronous gauge is used for calculation. We find that the numerical results for the resulting perturbations in the radiation era are the same, to many significant figures, as those got in the sudden approximation by application of the Lichnerowicz conditions [7] on a hypersurface of constant energy density, as formulated by Deruelle and Mukhanov [2]. This result serves to confirm the hypothesis (and its implementation) that the sudden phase transition happens on a hypersurface of constant energy density even where it is between entirely different types of physical states – in this case a scalar field and a standard type of perfect fluid.

In the following section we outline the model, setting out the scalar potential. This is adjusted to give a period of power-law inflation, followed by a transition to zero scalar field, with the transition being mediated by a friction term. In section III we first derive the resulting equations for the reheating transition in which the scalar field matter of the universe is wholly converted into a radiation fluid and the universe enters the radiation era; secondly we derive the equations for the development of the density perturbations, arisen by earlier quantum fluctuations, through the gradual reheating transition. Section IV is mainly on the numerical solution of these equations to give the resulting radiation era parameters; for the perturbations we treat only wavelengths important for the observed fluctuations of the cosmic microwave background. We also derive the corresponding radiation era perturbation for the sudden transition approximation, so as to compare this with the result of the gradual transition. We show that the two results are the same, to extremely good accuracy. The computations are done in the synchronous gauge, but the comparison of results for the sudden and gradual transitions are done using gauge invariant variables [8]. We also include a discussion of the effect of our gradual transition mechanism on non-physical so called 'synchronous gauge modes' - eliminable by a coordinate transformation using the residual gauge freedom of synchronous gauges [1,8].

The metric tensor perturbations - gravitational waves - are a separate and much easier case and in IV C we show analytically that the sudden transition approximation is valid also in this case.

II. MODEL FOR INFLATION AND REHEATING

We choose a potential for the scalar field ϕ which gives constant power law inflation for large ϕ and which, in conjunction with a friction term, converts the scalar field energy into a perfect fluid as ϕ decreases. We take the fluid to be the usual one of the radiation era, so that the pressure and density are related by $p = \frac{1}{3}\rho$. The potential is given by

$$V = U \exp(\lambda\phi), \{\lambda > 0\}; \phi > \phi_A, \tag{1}$$

$$V = V_0[\exp(-2\nu\phi) - 2\exp(-\nu\phi) + 1]; \phi < \phi_A. \tag{2}$$

and is illustrated in the Figure for the particular case where $V_0 = 1, \nu = \sqrt{2}$ and $\lambda = .45$.

The constants U, V_0, λ, ν are adjusted so that $V, dV/d\phi$ are continuous at $\phi = \phi_A$. The curvature, $d^2V/d\phi^2$, changes sign at $\phi_B < \phi_A$ so that it is negative for $\phi < \phi_B$ where a friction term $f\sqrt{d^2V/d\phi^2}$, where f is an adjustable constant, is added to the equations as detailed below.

The scalar field decreases with time and in the power law inflation region, $\phi > \phi_A$, the cosmic scale factor

$$a(\tau) \propto (\tau_p - \tau)^p \propto t^q \tag{3}$$

where τ is conformal and t cosmic time. τ_p is a constant, $\lambda < \sqrt{2}$ and

$$p = 2/(\lambda^2 - 2), q = 2/\lambda^2. \tag{4}$$

For $\phi < \phi_A$ the field accelerates until some $\phi < \phi_B$ where the friction becomes large enough to slow it down; finally the field energy is all converted into fluid energy.

This model is just one realization of inflation followed by reheating and is designed to track coupled metric, scalar field and density perturbations through inflation to the end of reheating. These perturbations are supposed to arise from quantum fluctuations early in the inflationary era. They are accompanied by quantum fluctuations with graviton production giving rise to gravity waves which we shall likewise track through to the radiation era.

III. INFLATION AND REHEATING EQUATIONS

A. The unperturbed equations

We write the equations for the case where the radiation fluid and the scalar field are coexisting; the pure scalar field era and the radiation era are just special cases of these equations. For $\phi > \phi_A$ the model has an analytic solution but otherwise a numerically computed solution is necessary, even for this unperturbed case.

The Einstein equations are

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T^s_{\mu\nu} + \kappa T^h_{\mu\nu}, \tag{5}$$

$$\kappa = 8\pi G \tag{6}$$

the right hand side being the sum of the energy-momentum tensors of the scalar field and of the fluid, with density ρ and pressure $p = b\rho$ and 4-velocity u^μ:

$$T^s_{\mu\nu} = \phi_{,\mu}\phi_{,\nu} - g_{\mu\nu}[\frac{1}{2}g^{\alpha\beta}\phi_{,\alpha}\phi_{,\beta} + V(\phi)] \tag{7}$$

$$T^h_{\mu\nu} = pg_{\mu\nu} + (\rho + p)u_\mu u_\nu \tag{8}$$

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To solve these equations the scalar field equation is also requisite. To the usual scalar equation we add a friction term so that it becomes

$$d^2\phi/d\tau^2 + 2aH d\phi/d\tau + a^2 V_1 + a V_f d\phi/d\tau = 0 \tag{9}$$

where Hubble, $H = (da/d\tau)/a^2$ and we have used the notations

$$V_1 \equiv dV/d\phi, \tag{10}$$

$$V_f = f\sqrt{d^2 V(\phi)/d\phi^2}, \phi < \phi_B, V_f = 0, \phi > \phi_B. \tag{11}$$

From the Einstein equations we obtain

$$H^2 = \frac{\kappa}{3}[(\frac{d\phi}{d\tau})^2/2a^2 + V + \rho]. \tag{12}$$

The Bianchi identities from the Einstein equations combined with the scalar equation yield the fluid energy equation;

$$d\rho/d\tau = -3aH(p + \rho) + a^{-1}V_f(d\phi/d\tau)^2 \tag{13}$$

Eqs. (9,12,13) are sufficient to solve the model; we note that as ϕ decreases kinetic energy is taken from the scalar field and given to the fluid. The Einstein equations remain valid.

For calculation it is convenient to use

$$x \equiv \ln a \tag{14}$$

as evolution parameter instead of τ and we use the notation $' \equiv d/dx$. The equations of motion then become

$$\phi'' + \frac{H'}{H}\phi' + 3\phi' + V_1/H^2 + V_f\phi'/H = 0, \tag{15}$$

$$H^2 = \kappa(V + \rho)/(3 - \frac{\kappa}{2}\phi'^2), \tag{16}$$

$$\rho' = -4\rho + HV_f\phi'^2, \tag{17}$$

where in the last equation we have used $p = \frac{1}{3}\rho$ for the relativistic fluid. In these equations V, V_1, V_f are given functions of ϕ only, and the solution of the equations gives $\rho(x), \phi(x), H(x)$ as functions of $x = \ln a$ only. We can find the value of τ as a function of x and thus of a by

$$\tau = \int^x \frac{dx}{H}\exp(-x). \tag{18}$$

We note that the equations scale in the following sense: if we have a solution with a certain V_0, Eq.(2), with a certain $\phi(x), \rho(x), H(x)$ and c is any constant *then* $c^2 V_0, \phi, c^2\rho, cH$ is also a solution. This is convenient because if we have a solution with a particular V_0 then, by adjustment of c, we can get a solution appropriate to the era we wish to consider.

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In this region $\rho = V_f = 0, V = U \exp(\lambda\phi), V_1 = U\lambda\exp(\lambda\phi)$ and Eqs.(15,16) yield

$$\phi''/(1 - \frac{\kappa}{6}\phi'^2) + 3(\phi' + \lambda/\kappa) = 0. \tag{19}$$

The simplest solution is

$$\phi' = -\frac{\lambda}{\kappa}, H^2 = 2U\exp(\lambda\phi)/(6 - \frac{\lambda^2}{\kappa}). \tag{20}$$

Letting $\Lambda \equiv \lambda^2/\kappa$ it follows that $H \propto \exp(-\frac{\Lambda x}{2}), \tau - \tau_p \propto \exp[(\frac{\Lambda}{2} - 1)x], a \propto (\tau - \tau_p)^p, p \equiv 2/(\Lambda - 2), \tau_p = constant$. With $\Lambda < 2$ this is the well known power-law inflation due to an exponential potential.

<center>*2. $\phi < \phi_A$*</center>

In this region we need to use numerical compution to solve even the unperturbed equations whose solution we shall denote by ϕ_0, ρ_0. In our numerical computation the unperturbed solution and the perturbation to it are evolved simultaneously, and we proceed now to the formalism for this latter.

B. The perturbed equations

Perturbations to the energy-momentum tensor density are coupled to scalar perturbations of the metric tensor. For these we work in a synchronous coordinate system and follow in the main the formalism of Grishchuk [1]. The metric is

$$ds^2 = -a^2 d\tau^2 + a^2[\delta_{ij} + h_{ij}(x)]dx^i dx^j. \tag{21}$$

$h_{ij}(x)$ are the relevant perturbations which can be expressed as a linear superposition for the various wave numbers $\mathbf{k}$ of

$$h_{ij}(\mathbf{k}, x) = h(\mathbf{k}, \tau)Q\delta_{ij} + h_l(\mathbf{k}, \tau)k^{-2}Q_{,i,j}, \tag{22}$$

where Q is a superposition of the spatial wave function solutions $\exp(\pm i\mathbf{k}.\mathbf{x})$ so that $Q_{,i,j} = -k_i k_j Q$. The total perturbation can be found by solving for h, h_l and this we shall do. We write the $\mathbf{k}$-components of the *perturbation* to the energy-momentum tensor as

$$T^0_{k0} = \bar{\rho}_1 Q, T^0_{ki} = -T^i_{k0} = a^{-2}\alpha\bar{\xi}'Q_{,i}, \tag{23}$$

$$T^j_{ki} = (\bar{p}_1 + \bar{p}_l)Q\delta^j_i + k^{-2}\bar{p}_l Q^j_i. \tag{24}$$

where $\alpha \equiv dx/d\tau = \frac{da}{d\tau}/a = aH$, and where $\bar{\rho}_1$ and $\bar{p}_1, \bar{p}_l$ are density and pressure perturbation components appropriate to $\mathbf{k}$. Then the perturbed Einstein equations where the variables are functions of $x = \ln a$ are

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$$3h' + (k/\alpha)^2 h - h'_l = \kappa(a/\alpha)^2 \bar{\rho}_1, \tag{25}$$

$$h' = \kappa \bar{\xi}' \tag{26}$$

$$-h'' - \frac{H'}{H}h' - 3h' = \kappa(a/\alpha)^2 \bar{p}_1, \tag{27}$$

$$\frac{1}{2}[h''_l + \frac{H'}{H}h'_l + 3h'_l - (k/\alpha)^2 h] = \kappa(a/\alpha)^2 \bar{p}_l. \tag{28}$$

In what follows we shall denote the **k**-component perturbation to the scalar field by ϕ_1 and the **k**-component fluid perturbations by ρ_1, p_1, ξ' and p_l. For the perfect fluid, which we assume, $p_l = 0$ and the corresponding quantity vanishes for the scalar field; so $\bar{p}_l = 0$. The right hand sides of Eqs.(25-28) are given by:

$$(a/\alpha)^2 \bar{\rho}_1 = \phi'_0 \phi'_1 + (\phi_1 V_1 + \rho_1)/H^2, \tag{29}$$

$$\bar{\xi}' = \xi' - \phi'_0 \phi_1, \tag{30}$$

$$(a/\alpha)^2 \bar{p}_1 = \phi'_0 \phi'_1 - (\phi_1 V_1 - \frac{1}{3}\rho_1)/H^2, \tag{31}$$

$$(a/\alpha)^2 \bar{p}_l = 0. \tag{32}$$

Inserting these into the preceeding equations we have a set of differential equations capable of solution when we have added the following perturbed scalar field equation;

$$\phi''_1 + \frac{H'}{H}\phi'_1 + 3\phi'_1 + \frac{\phi_1}{H^2}\frac{d^2V}{d\phi^2} + [\frac{k}{aH}]^2\phi_1 + \frac{1}{2}(3h' - h'_l)\phi'_0 = -[\frac{dV_f}{d\phi}\phi'_0 \phi_1 + V_f \phi'_1]/H \tag{33}$$

This equation combined with the (perturbed) Bianchi identities yields the perturbed fluid energy equation as:

$$\rho'_1 + 4\rho_1 - (k/a)^2 \xi' + \frac{2}{3}\rho_0(3h' - h'_l) = [\frac{dV_f}{d\phi}\phi'^2_0 \phi_1 + 2V_f \phi'_0 \phi'_1]H \tag{34}$$

As in the unperturbed case the physical assumption lies in the addition of the friction term to the scalar equation; the Einstein equations are preserved leading to the perturbed fluid energy equation, including the appropriate friction term which balances the one in the scalar equation.

IV. GRADUAL AND SUDDEN TRANSITIONS

A. Inflationary era

The initial conditions for solving the equations in the transition region, $\phi_A > \phi > 0$, are given by the solution at the end, $\phi = \phi_A$, of the power law inflation in the region $\phi > \phi_A$. In that region we already have the unperturbed solution in III A 1 and shall now use well-known analytic methods to get the perturbed solution.

We shall use dotted quantities to denote differentiation with respect to τ; for example

$$\dot{h} \equiv dh/d\tau \tag{35}$$

From Eqs.(26,30)

$$\kappa\phi_1 = -\dot{h}/\dot{\phi}_0 \tag{36}$$

It can be shown that for the scalar field only case [1]

$$\ddot{\mu} + \mu[k^2 - (a\sqrt{\gamma})^{\cdot\cdot}/(a\sqrt{\gamma})] = 0 \tag{37}$$

$$\mu \equiv \frac{a}{\alpha\sqrt{\gamma}}(\dot{h} + \alpha\gamma h), \tag{38}$$

$$\gamma \equiv 1 - \dot{\alpha}/\alpha^2, \tag{39}$$

where, as defined in III B, $\alpha = \dot{a}/a$. The scale factor is given by Eq.(3) so that $\gamma = (1+p)/p = $ constant.

Integration of Eq.(38) gives h as

$$h = \frac{\alpha}{a}\{\sqrt{\gamma}\int^{\tau}\mu(k,\tau)d\tau + C_i\}, \tag{40}$$

where C_i is an integration constant subsuming, when γ is constant, the lower limit of the integration. $h = \frac{\alpha}{a}C_i$ is a solution of $\mu = 0$ which can be eliminated by a coordinate transformation allowed by the residual gauge freedom of the synchronous gauge [1]; it gives zero contribution to gauge invariant variables [8,9]. We drop this term here but we shall revisit it later in conjunction with the results.

All the above development was non-quantum mechanical. We shall now briefly remark on the quantum basis of the phenomena. We denote the corresponding quantum field theory quantities by a tilde:

$$\tilde{h} = \frac{\alpha}{a}\sqrt{\gamma}\int^{\tau}\tilde{\mu}(k,\tau)d\tau \tag{41}$$

$$\tilde{\mu} = N\int\frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2k}}[c_k\mu_1(k,\tau)\exp(i\mathbf{k}.\mathbf{x}) + h.c.] \tag{42}$$

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where c_k is a quantum annihilation operator, $[c_k, c_{k'}^{\dagger}] = \delta^3(\mathbf{k} - \mathbf{k}')$, and $\mu_1(y), y = |k(\tau_p - \tau)|$, is that solution of the Bessel equation (37) such that

$$\mu_1(y) \to e^{-iy}, y \to \infty \tag{43}$$

thus corresponding for large k to the usual mode function of quantum mechanical plane waves.

N is a normalization factor whose determination gives the absolute magnitude of the observed density perturbations (in terms of the model parameters) given the assumption that these come from a primordial vacuum with zero quantum occupation number. N has been determined as [1,9]

$$N = \sqrt{2\kappa} = \sqrt{16\pi G}. \tag{44}$$

For this paper, which is mainly about sudden transition versus gradual transition amplitudes, we do not need to discuss the quantum aspects further.

We can now proceed with the evaluation of the metric scalar components at the end of inflation. The exact expression for the solution μ_1 is, with $n = \frac{1}{2} - p$,

$$\mu_1(y) = \sqrt{\frac{\pi y}{2}}(J_n - iY_n)\exp[-i(\frac{1}{2}n\pi + \frac{1}{4}\pi)] \tag{45}$$

Exact power-law inflation ends and the transition region begins at $\phi = \phi_A$; other variables at that surface,A , we denote by the suffix A. For example: $y_A = |k(\tau_p - \tau_A)|$. For small enough values of y_A we can expand μ given by Eq.(45) in an ascending power series and just consider the first one or two terms. Our first interest is in values of k of importance in COBE and other observations. These are of the order of magnitude (the relevant meaning here is being within a few factors of ten) of

$$k_1 \equiv a_1'/a_1 = a_1 H_1 \tag{46}$$

where a_1, H_1 denote the values of the scale factor and Hubble at the time, τ_1, when the matter era begins. Thus for such values of k the values of y_A are of order $10^{-n}, n > 10$. Then we expand μ in a power series with leading terms

$$\mu_1(y) = M(p)[y^p - \frac{y^{p+2}}{2(2p+1)}.....], \tag{47}$$

$$M(p) \equiv -2^{-p}\Gamma(\frac{1}{2} - p)\exp(-i(1-p)/2)/\sqrt{\pi}. \tag{48}$$

The corresponding expansion of h from Eq.(40), with $C_i = 0$, is

$$h = (\sqrt{\gamma}a)^{-1}M(p)[y^p - \frac{(p+1)y^{p+2}}{2(2p+1)(p+3)}.....] \tag{49}$$

We need these expressions at $y = y_A$ to find the initial values for the numerical solution of the coupled differential equations in h, h_l, ϕ_1, ρ_1. Noting that the first term in the expansion of h

$$\Sigma \equiv (\sqrt{\gamma}a)^{-1}M(p)y^p \tag{50}$$

is a constant, we find to first order in y_A^2 the initial values

$$h = \Sigma + y_A^2 P_1 \Sigma \tag{51}$$

$$h' = \frac{2}{p}y_A^2 P_1 \Sigma \tag{52}$$

$$h_l' = \frac{1}{p(2p+1)}y_A^2 \Sigma \tag{53}$$

$$\phi_1 = \frac{2}{p\lambda}y_A^2 P_1 \Sigma \tag{54}$$

$$\phi_1' = \frac{2}{p}\phi_1 \tag{55}$$

$$\rho_1' = 0 \tag{56}$$

where

$$P_1 = -(p+1)/[2(p+3)(2p+1)] \tag{57}$$

B. Reheating and the radiation era

In Eqs.(25-28) h occurs only when multiplied by $(k/\alpha)^2 \equiv y^2/p^2$. This and the initial conditions above lead us to rewrite the equations in terms of renormalized variables which are of order zero in y^2. These are:
$\hat{h} = (\alpha_A/k)^2(h - \Sigma)/\Sigma$, $\hat{h}_l' = (\alpha/k)^2 h_l'/\Sigma$, $\hat{\phi}_1 = (\alpha_A/k)^2\phi/\Sigma$, $\hat{\rho}_1 = (\alpha_A/k)^2\rho/\Sigma$, $\hat{\xi}_1 = (\alpha_A/k)^2\xi/\Sigma$.
Except for h_l' the renormalization is by the same constant factor coinciding with the factor for h_l' at A. The initial conditions for the hatted variables are given by the replacement $y_A^2\Sigma \to p^2$ on the right hand sides of Eqs.(51-56). The hatted versions of the last two Einstein equations are

$$\hat{h}'' + (\frac{H'}{H} + 3)\hat{h}' = -\kappa[\phi_0'\hat{\phi}_1' - (\hat{\phi}_1 V_1 - \frac{1}{3}\hat{\rho}_1)/H^2], \tag{58}$$

$$(\hat{h}_l')' + (1 - \frac{H'}{H})\hat{h}_l' - 1 = 0 \tag{59}$$

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We shall supplement Eqs.(58,59) by Eqs.(60,61) below, which are the hatted versions of Eqs.(33,34). We shall omit the explicit k^2 terms because these are of order $(k/\alpha)^2 \equiv y^2/p^2$ smaller than the other terms in the equation for both cases. This is a consistent, controlled approximation, extremely good for COBE and related observations.

$$\hat{\phi}_1'' + (\frac{H'}{H} + 3)\hat{\phi}_1' + \frac{\hat{\phi}_1}{H^2}\frac{d^2V}{d\phi^2} + \frac{1}{2}[3\hat{h}' - ((aH)_A/(aH))^2\hat{h}_l']\phi_0' = -[\frac{dV_f}{d\phi}\phi_0'\hat{\phi}_1 + V_f\hat{\phi}_1']/H \quad (60)$$

$$\hat{\rho}_1' + 4\hat{\rho}_1 + \frac{2}{3}\rho_0[3\hat{h}' - ((aH)_A/(aH))^2\hat{h}_l'] = [\frac{dV_f}{d\phi}\phi_0'^2\hat{\phi}_1 + 2V_f\phi_0'\hat{\phi}_1']H \quad (61)$$

Eqs.(58)-(61), with the initial conditions specified above, have to be solved for the perturbations. It is necessary to know the unperturbed solutions ϕ_0, ρ_0 and H which enter the equations; these are given by eqs.(15)- (17) and their initial conditions at the beginning of reheating are given in III A 1. We evolve all the variables, both the perturbed and the unperturbed, together as a set of coupled differential equations with evolution parameter $x = \ln a$. The end of the transition period and the beginning of the radiation era are signalled by ϕ_0 arriving at a negligable value; ϕ_1 accompanies ϕ_0 into oblivion.

Our objective is to compare the radiation era amplitudes resulting from this gradual transition to those resulting from a sudden transition.

1. Sudden transition to the radiation era

For this we use the matching conditions given by Deruelle and Mukhanov [2]; conditions (their eqs. 4.5a,b) in the longitudinal gauge can equivalently be given in terms of the gauge invariant variable Φ. They are that Φ and Γ are continuous across the matching surface $\tau = \tau_2$:

$$\Phi_+ \equiv \Phi(\tau_{2+}) = \Phi(\tau_{2-}) \equiv \Phi_- \quad (62)$$

$$\Gamma_+ \equiv \Gamma(\tau_{2+}) = \Gamma(\tau_{2-}) \equiv \Gamma_- \quad (63)$$

where

$$\Gamma \equiv (\Phi' + \Phi + \epsilon^2\Phi)/\gamma, \quad (64)$$

$$\epsilon = k/(\alpha\sqrt{3}). \quad (65)$$

Additionally, more obviously, a and $\alpha = \dot{a}/a = aH$ must also be continuous across the surface.

Density perturbations can be described by the mode function ν, corresponding to the μ_1 of the inflation era, such that [1,9]

$$\ddot{\nu} + \frac{1}{3}k^2\nu = 0. \quad (66)$$

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In terms of the synchronous gauge variables

$$\nu \equiv \frac{a}{\alpha}(\dot{h} + \alpha\gamma h) \equiv a(h' + \gamma h) \tag{67}$$

and the gauge invariant variable Φ is given by [8]

$$\Phi = h - (\alpha/k)^2 h'_l \tag{68}$$

so that from Eqs.(25-28)

$$\Phi = (\nu - \nu')/a\epsilon^2 \tag{69}$$

Non-physical synchronous gauge modes (that is of the form $h = \alpha/a \times constant$) give zero contribution to both ν and Φ. We write the solution of Eq.(66) as

$$\nu = C\cos(k(\tau - \tau_2)/\sqrt{3}) - S\sin(k(\tau - \tau_2)/\sqrt{3}). \tag{70}$$

so that

$$\Phi_+ = (\epsilon_2 S + C)/(a_2\epsilon_2^2), \tag{71}$$

$$\Gamma_+ = -[C(1 - \epsilon_2^2) + \epsilon_2 S(1 - \frac{1}{2}\epsilon_2^2)]/(a_2\epsilon_2^2). \tag{72}$$

From the continuity of Φ and Γ across the interface these expressions enable the determination of C, S and thus the density perturbation in the radiation era. So we need the values of Φ_- and Γ_- from the inflationary era solutions. From Eqs.(51, 53,68), interpreted now as being at the surface $\tau = \tau_2$, we find that to lowest order in $(k/\alpha)^2$

$$\Phi_- = \frac{p+1}{2p+1}\Sigma; \Gamma_- = \frac{p}{2p+1}\Sigma. \tag{73}$$

Solving Eqs.(62,63) straightforwardly gives to lowest order

$$C = -\epsilon_2 S = 2a_2\Sigma. \tag{74}$$

For k relevant to CMBR fluctuation observations ϵ_2 is of order 10^{-n} where $n > 10$ and thus away from the interface the term in S dominates the radiation era density perturbations.

2. Gradual reheating transition

For our reheating model, or any such continuous transition, no question of sophisticated implementation of Lichnerowicz conditions at an interface arises. There is no interface; the physical variables arising from the solutions of the equations of motion are all continuous through space-time.

We shall consider the case where the transition ends at the same surface, $\tau = \tau_2$, as that of IV B 1; ϕ is negligable for $\tau \geq \tau_2$ and the system is in the radiation era. The transition

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begins at the end of pure power law inflation, $\tau = \tau_A$. We note here that Σ of Eq.(50) is a constant of the power-law inflation era. Below we shall refer to a Σ evaluated at τ_A at the end of power-law inflation at the beginning of the gradual transition; while in IV B 1 we referred to Σ evaluated at that appropriate (sudden) end of power-law inflation, τ_2; these have the same name and are indeed the same numbers.

The radiation era analysis is of course precisely the same as in IV B 1, with Eqs.(68,69) giving

$$3\Phi + \Phi' = -\frac{3}{ak^2}\ddot{\nu} = \frac{1}{a}\nu; \tag{75}$$

so at the end of the transition, $\tau = \tau_2$, which is the beginning of the radiation era

$$C/a_2 = 3\Phi_2 + \Phi'_2, \tag{76}$$

$$(C + \epsilon_2 S)/(a_2 \epsilon_2^2) = \Phi_2. \tag{77}$$

Putting into the RHS of the last two equations the computed values at the end of the transition we can find C and S. In fact, because of the very small magnitude of ϵ_2, the last equation implies that to lowest order in ϵ_2

$$\epsilon_2 S = -C \tag{78}$$

since otherwise Φ_2 would take a large value unattainable in the transition. The result we have arrived at is that for the continuous case the density perturbation in the radiation era is determined by the value of $3\Phi_2 + \Phi'_2$ at the end of the gradual transition.

By numerical integration of Eqs.(58-61) with the stated initial conditions we determine Φ_2 and Φ'_2 in terms of Σ using Eq.(68). We can then compare the results for S and C, determining the radiation era perturbations, from the sudden and gradual transitions.

3. Comparison of sudden with gradual transition

For the sudden transition we have, Eq.(74),

$$-\epsilon_2 S/a_2 \Sigma = C/a_2 \Sigma = 2 \tag{79}$$

For the gradual transition we denote

$$result \equiv (3\Phi_2 + \Phi'_2)/\Sigma \tag{80}$$

so that

$$-\epsilon_2 S/a_2 \Sigma = C/a_2 \Sigma = result \tag{81}$$

Thus if $result = 2$ the two transitions agree.

We have calculated the gradual transition for two values of the inflation power-law exponent, p, each for a number of values of the friction coefficient, f, with the results shown

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in the Tables. It may be noted that for the larger values of f ,($f > 1$) quoted the system grinds to a halt at the bottom of the potential well with no significant oscillations; this of course does not resemble any usually envisaged reheat mechanisms [10]. For smaller f oscillations occur and it is found, as we would expect, that their number increases rapidly as f decreases. Our numerical computation methods fail for f significantly smaller than the last value in the Tables, $f = .01$.

As stated previously the transition is deemed to have come to an end when the value of ϕ_0 is sufficiently small. There is not a precise criterion and we display results for various values of $\phi_0(end)$. We see that always $result = 2$ to very high accuracy, in agreement with the sudden transition.

This establishes the agreement (for appropriate wavelengths) in the following sense: whether there is a sudden transition at red-shift z_2 or our type of gradual transition ending at z_2, the resulting radiation era scalar perturbation is the same.

4. Synchronous gauge mode

As set out in Eqs.(36)-(40) there is in the inflationary era (as in other eras) a 'synchronous gauge mode' given by

$$h = \frac{\alpha}{a}C_i \tag{82}$$

where C_i is a constant. This mode is non-physical as it contributes zero to the gauge invariant variable Φ, given by Eq.(68), as well as to other gauge invariant variables, and also it can be eliminated by a coordinate transformation making use of the residual gauge freedom of synchronous gauges.

Whole these remarks justify our ommission of this term, we have investigated further so as to provide a comment on the mechanism of energy transfer inthe gradual transition of our model. Our theory is a linear perturbation theory , so we can completely investigate the consequences of a synchronous gauge mode separately from the physical modes previously considered by inserting solely Eq.(82) and its derivative and

$$h'_l = (k/\alpha)^2 \frac{\alpha}{a}C_i \tag{83}$$

as the initial conditions in Eqs.(58)-(61). These correspond to zero Φ in the inflation era. For a perfect, that is coordinate independent, perturbed transition mechanism in Eqs.(60) and (61) the resulting Φ in the radiation era would be zero. To give an idea of the magnitudes involved we can put, formally, $(\frac{a}{a})_A C_i = \Sigma$ at the beginning of the transition - that is as if this quantity began the transition with the same magnitude as the physical mode previously considered. We illustrate by giving two results, corresponding to the first and last lines of Table I, for the value of Φ at the end of the transition:

$$f = 4 \Rightarrow \Phi_2 = \Sigma \times 10^{-3}, \tag{84}$$

$$f = .01 \Rightarrow \Phi_2 = 4.36\Sigma \times 10^{-6}. \tag{85}$$

The results for the physical mode were 2Σ for whatever f. Eqs. (84) and (84) illustrate firstly that there is a coordinate dependence in the transition mechanism when applied to the non-physical, coordinate dependent, perturbative mode and secondly that it decreases with f.

It is probable that other transition mechanisms expressible as terms in the Lagrangian, such as some of those which have been invoked to investigate particle production in the reheat phase [11], would transmit only coordinate independent information into coordinate independent variables.However investigation of these would involve different formalisms and calculations. We would not expect any different answers for the physical modes.

C. Gravitational waves

Tensor perturbations are very much simpler to treat than scalar perturbations. After their generation by quantum fluctuations their development is governed solely by the cosmic scale factor a. Corresponding to Eq.(42) we have in the power-law inflationary era

$$\tilde{h}_{ij} = \sqrt{16\pi G} \sum_{\lambda=1}^{2} \int \frac{d^3k}{(2\pi)^{\frac{3}{2}}\sqrt{2k}a(\tau)}[a_{\lambda k}\epsilon_{ij}^{\lambda}(k)\mu_1(k,\tau)\exp(i\mathbf{k}.\mathbf{x}) + h.c.], \qquad (86)$$

where $a_{\lambda k}$ is the annihilation operator for the graviton with polarization λ and wave number $\mathbf{k}$, and the polarization tensor satisfies $\sum_{i,j}\epsilon_{ij}^{\lambda}(k)\epsilon_{ij}^{\lambda'}(k) = 2\delta_{\lambda\lambda'}$. μ_1 given by Eq.(47) is a solution, appropriate to the quantum mechanics as in Eq.(43), of the tensor mode equation

$$\ddot{\mu} + \mu(k^2 - \ddot{a}/a) = 0. \qquad (87)$$

This holds for any cosmic era, whatever may be the dynamics responsible for the particular form of $a(\tau)$ and in the radiation era, beginning at $a = a_2$, which defines $\tau = \tau_2$, we can write its solution as

$$\mu = G_+ \cos(k(\tau - \tau_2)) - G_- \sin(k(\tau - \tau_2)), \qquad (88)$$

where G_+ and G_- are constants to be determined by continuity, as set out below.

1. Sudden transition

If we assume a sudden transition from inflation to the radiation era at $a = a_2$ the matching conditions that determine G_+ and G_- are that μ and $\dot{\mu}$ (equivalently μ') must be continuous across the interface; the continuity of a and α is also a condition but we already hold these enforced as in III B [2,12]. Up to the second term of the power series expansion μ is given by Eq.(47) and μ' by

$$\mu_1'(y) = M(p)[y^p - \frac{(p+2)y^{p+2}}{2p(2p+1)}], \qquad (89)$$

and the required continuity gives

$$G_+ = M(p)y_2^p[1 - \frac{y_2^2}{2(2p+1)}], \tag{90}$$

$$G_- = pM(p)y_2^{p-1}[1 - \frac{(p+2)y_2^2}{2p(2p+1)}]. \tag{91}$$

Due to the extreme smallness of y_2 for waves of observational interest G_- dominates and to lowest order

$$G_+ = 0, \tag{92}$$

$$G_- = pM(p)y_2^{p-1}. \tag{93}$$

2. Comparison of gradual with sudden transition

At the end of power-law inflation, beginning the gradual transition

$$\mu(\tau_A) = M(p)y_A^p[1 - \frac{y_A^2}{2(2p+1)}], \tag{94}$$

$$\mu'(\tau_A) = M(p)y_A^p[1 - \frac{(p+2)y_A^2}{2p(2p+1)}]. \tag{95}$$

Also we shall below use the fact (see Eq.(50)) that

$$\Sigma_G \equiv= M(p)y^p/a = \sqrt{\gamma}\Sigma \tag{96}$$

is a constant of the inflationary era. It is convenient to consider the function $\lambda \equiv \mu/a$ so that from Eq.(87)

$$\ddot{\lambda} + 2\dot{\lambda}\dot{a}/a + k^2\lambda = 0 \tag{97}$$

Generally the solution is not trivial, but working to lowest order, putting $k^2 = 0$, we get

$$a^2\dot{\lambda} = constant \tag{98}$$

and, again to lowest order, at the end of the power-law inflation

$$\lambda = M(p)y^p/a \Rightarrow \dot{\lambda}_A = 0. \tag{99}$$

Then λ is constant throughout the transition, $\mu \propto a$, as it also is in the inflationary phase, though for different $a(\tau)$; in both cases $\lambda = \Sigma_G$. This implies that, to lowest order, where μ and μ_1 are the gradual transition and inflationary mode functions respectively:

$$\mu(a_2) = \mu'(a_2) = \mu_1(a_2) = \mu_1'(a_2) \tag{100}$$

since $a' = a$. Thus Eqs.(92,93) hold as in the sudden transition case.

This establishes the same result for tensor perturbations as was established for scalar perturbations, and in the sense stated at the end of section IV B 3. For scalar perturbations it was likewise justifiable to neglect terms of order $(k/\alpha)^2$ in certain equations for waves with k relevant to current observations. We may note that this easy way of finding the result in the tensor case is not only due to the relative simplicity of the governing equation (87) but also to the assumption that the inflation is power-law. However we would expect it to hold more generally.

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V. DISCUSSION

As the transition proceeds both the unperturbed, ϕ_0, and the perturbed, ϕ_1, scalar fields diminish so that at some sufficiently small value we can use the perfect fluid model of the radiation era. In the latter part of the radiation era, as the the matter era is approached, the amplitude of the density perturbation is specified by the single quantity $result \equiv 3\Phi + \Phi' = 3\Phi + (aH)^{-1}d\Phi/d\tau$, evaluated at the end of the transition.

An extension of the length of the transition, specified in the Tables by the number of e-foldings, can always give values of $|\phi_0|$ as small as we please until the computation runs out of numerical accuracy. We have chosen the varying examples in the tables to demonstrate that the lack of precision in the $|\phi_0|$ criterion is not important in the sense that $result \equiv (3\Phi_2 + \Phi_2')/\Sigma$ is very near to 2 for many different 'end' values of $|\phi_0|$.

Though Φ_2/Σ and Φ_2'/Σ vary in the third decimal place the $result$ is much more constant, and we can understand this: In the radiation era by Eqs.(75,70) $(3\Phi + \Phi')/\Sigma = -\epsilon S/a\Sigma + O(\epsilon^2)$ and the first term is constant since $a \propto \tau$ and $\epsilon \propto \tau$ and an equivalent result would apply towards the end of the transition region. Thus for the above calculations with small k the value of $3\Phi + \Phi'$ is very constant over many e-foldings of the cosmic scale factor a after the scalar field has decreased in magnitude by more than about 4 powers of 10. This is a property of the early part of the radiation era and not a property of the particular transition mechanism adopted.

Having concurred with some other authors [2–5,13], through different considerations, that transition period detail has no influence on density perturbations important for the CMBR fluctuations, we may ask for what wave numbers k the detail is going to be important. The answer is likely to vary considerably with the assumed physics of the early universe. So we can only give an answer for power-law inflation followed by reheating. For illustration it is simplest to consider the case of gravitational waves as in IV C. To get the gradual-sudden equivalence there required that $p^2 k^2/\alpha^2 \ll 1$ throughout the transition region. We can conclude that for $k/k_1 \approx 10^{23} h^{-.5}$ the sudden result is still reasonably good; here h is the Hubble parameter. k_1, given by Eq.(46, for redshift $z_1 = 10^{3.3}$, being the value around which we did our tabulated calculations above, specifies the region of wave numbers of importance in CMBR fluctuations. Translated into energy terms this means that gravitational waves with $k < 10^{-11} h^{.5} eV$ would not be affected by reheating details. A calculation with density perturbations gives similar results.

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TABLE I. $result \equiv (3\Phi_2 + \Phi_2')/\Sigma$ at the 'end' (see text) of the gradual transition for values of the friction coefficient, f; R_ϕ is the ratio of the unperturbed scalar field value, ϕ_0, at the transition 'end' to that at the beginning; $x(end)$ gives the number of e-foldings of the transition. Conformal time inflation power $p = -1.113$.

f	$x(end)$	$|R_\phi|$	Φ_2/Σ	Φ_2'/Σ	$|result - 2|$
4.0	3.2	0.29×10^{-10}	0.66162	0.01515	$< 10^{-8}$
2.0	3.0	0.20×10^{-23}	0.66262	0.01214	$< 10^{-8}$
1.0	3.0	$0..30 \times 10^{-18}$	0.66343	0.00971	$< 10^{-8}$
0.5	3.0	0.34×10^{-10}	0.66337	0.00990	$< 10^{-8}$
0.1	3.5	0.18×10^{-6}	0.66450	0.00650	1.1×10^{-8}
0.05	4.0	0.14×10^{-6}	0.66534	0.00399	$< 10^{-8}$
0.01	5.0	0.54×10^{-7}	0.66540	0.00380	$< 10^{-8}$

TABLE II. $result \equiv (3\Phi_2 + \Phi_2')/\Sigma$ at the 'end' (see text) of the gradual transition for values of the friction coefficient, f; R_ϕ is the ratio of the unperturbed scalar field value, ϕ_0, at the transition 'end' to that at the beginning; $x(end)$ gives the number of e-foldings of the transition. Conformal time inflation power $p = -1.05$.

f	$x(end)$	$|R_\phi|$	Φ_2/Σ	Φ_2'/Σ	$|result - 2|$
4.0	4.0	0.61×10^{-10}	0.66236	0.01293	$< 10^{-8}$
2.0	4.0	0.21×10^{-17}	0.66476	0.00572	$< 10^{-8}$
1.0	3.0	0.26×10^{-4}	0.63595	0.09215	2.5×10^{-8}
0.5	4.0	0.11×10^{-15}	0.66511	0.00467	$< 10^{-8}$
0.1	4.5	0.50×10^{-9}	0.66566	0.00303	$< 10^{-8}$
0.05	5.0	0.48×10^{-10}	0.66605	0.00185	$< 10^{-8}$
0.01	6.0	0.15×10^{-10}	0.66608	0.00176	$< 10^{-8}$